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FREE CONVECTION FROM A POINT HEAT SOURCE IN A STRATIFIED FLUID*

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A non-stationary problem of free convection from a point heat source in a stratified fluid is considered. The system of equations is reduced to a single equation for a special scalar function which determines the velocity field, and the temperature and salinity distribution. Relations are found connecting the spatial and temporal scales of the phenomenon with the parameters of the medium and the intensity of the heat source. The magnitude of the critical source intensity at which the fluid begins to move in a jet-flow mode is established.

The structure of convective flows above the heat sources depends, in the stratified media, essentially on the nature of the stratification /1/ which may be caused by a change in the temperature of the medium /2, 3/ or its salinity /4-7/, and by the form of the heat source. When a temperature gradient exists within the medium, an ascending jet forms above the point source, mushrooming outwards near the horizon of the hydrostatic equilibrium. In the case of a fluid with salinity gradient, the jet is surrounded by a sheet of descending salty fluid, and a regular system of annular convective cells is formed around it /1/.

The height of the stationary jet computed in /2, 3/ on the basis of conservative laws agrees with experiment. However, this approach does not enable the temperature and velocity distribution over the whole space to be found and does not enable the problem of determining the flow to be investigated. A stationary solution of the linearized convection equations /8/ does not correspond to detail to the observed flow pattern /1, 5-7/. In this connection the study of the non-linear, non-stationary convection equations is of interest.

The purpose of this paper is to construct a non-linear, non-stationary free convection equation above a point heat source, and to analyse the scales of the resulting structure and the critical conditions under which the flow pattern changes.

1. Formulation of the problem.

A system of non-stationary, non-linear equations of thermoconcentration convection in a stratified fluid in a cylindrical system of coordinates with a point heat source of strength P lying at the origin of coordinates and with the vertical z axis directed opposite to the force of gravity vector g, has the form

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p+\nu\rho\Delta\mathbf{u}+\frac{1}{3}\nu\rho\nabla(\nabla\cdot\mathbf{u})+ \tag{1.1}$$
$$\rho_0(\beta S'-\alpha T')g,\quad \frac{\partial S}{\partial t}+\nabla\cdot(S\mathbf{u})=k_S\Delta S$$
$$\frac{\partial T}{\partial t}+\nabla\cdot(T\mathbf{u})=\chi\Delta T+\frac{P}{c_p\rho_0}\frac{\delta(z)\delta(r)}{2\pi r}\theta(t)$$
$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=-\frac{\alpha P}{c_p}\frac{\delta(z)\delta(r)}{2\pi r}\theta(t)$$
$$\rho=\rho_0(1+\beta S-\alpha T),\quad S=S_0(z)+S'$$
$$T=T_0(z)+T',\quad S_0(z)=S_0\left(1-\frac{z}{\beta S_0\Lambda_S}\right)$$
$$T_0(z)=T_0\left(1+\frac{z}{\alpha T_0\Lambda_T}\right)$$

Here $\mathbf{u}$ is the velocity of the medium, p is the pressure behind the residue of the hydrostatic pressure, $S, S(z), S$ are the total, stratified and additional salinity, $T, T(z), T'$ are the total, stratified and excess temperature, S_0, T_0, ρ_0 are the salinity, temperature

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and density of the medium at the level $z=0$, ρ is the density of the medium, $\beta, \alpha, \chi, \nu, k_S$ are the coefficients of saline and temperature expansions, thermal diffusivity, kinematic viscosity and salt diffusion, c_p is the heat capacity of the medium at constant pressure, and Λ_S, Λ_T are the scales of the salt and temperature stratification. The initial and boundary conditions for the functions $\mathbf{u}, p, S', T'$ are homogeneous.

Eqs.(1.1) imply that the condition that the velocity field is solenoidal

$$\nabla \cdot \mathbf{u} = -\Delta h, \quad h = \beta k_S S' - \alpha \chi T' \tag{1.2}$$

does not hold. A velocity field which is axisymmetric, admits of a representation of the form /9/

$$\begin{aligned} &\mathbf{u} = \mathbf{v} + \mathbf{w}, \quad \mathbf{v} = -\nabla h, \quad w_r = -\frac{\partial \varphi}{\partial r}, \quad w_z = -\frac{\partial \psi}{\partial z} \\ &\Delta_r \varphi + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad \Delta_r = r^{-1} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) \end{aligned} \tag{1.3}$$

where w_r, w_z is the radial and vertical component of the solenoidal part of the velocity, and φ, ψ are unknown functions of the coordinates and time.

The components of the total velocity vector can be written in the form

$$u_r = -\frac{\partial h}{\partial r} - \frac{\partial^2 f}{\partial r \partial z}, \quad u_z = -\frac{\partial h}{\partial z} + \Delta_r f, \quad f = \int_0^z \varphi(r, \xi, t)\, d\xi \tag{1.4}$$

Eqs.(1.1) contain non-linear terms which play an important part in forming the flow. Using the fact that the solenoidal part of the velocity is the main contributor towards the transfer, we shall use the following approximation:

$$(\mathbf{u} \cdot \nabla) A \approx (\mathbf{w} \cdot \nabla) A$$

where A is either a scalar (S', T', ρ), or a vector $(\mathbf{u})$ quantity. Using the Navier-Stokes equations and expressions (1.4), we obtain the function

$$\begin{aligned} &F(r, z, t) = \frac{\partial^2 f}{\partial r \partial z} \frac{\partial}{\partial r} \Delta_r f - \frac{1}{2} \frac{\partial}{\partial z} [\Delta_r f]^2 + \frac{1}{2} \frac{\partial}{\partial z} \left[\frac{\partial^2 f}{\partial r \partial z}\right]^2 - \\ &\frac{\partial}{\partial z} \int_0^r \Delta_R f(R, z, t) \frac{\partial}{\partial R} \frac{\partial^2 f(R, z, t)}{\partial z^2}\, dR = g(\beta S' - \alpha T') + \Delta\left(\frac{\partial}{\partial t} - \nu \Delta\right) f \end{aligned} \tag{1.5}$$

Using relations (1.2) and (1.5), we can eliminate S' and T' from the system of initial equations, thus reducing it to the system

$$\begin{aligned} &Dh = \frac{1}{g}\left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right)(F - D_\nu f) - \left(\frac{1}{\Lambda_S} + \frac{1}{\Lambda_T}\right) \Delta_r f + \\ &Q \frac{\delta(z)\delta(r)}{2\pi r} \theta(t), \quad \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla - \chi \Delta\right) h + \frac{\chi - k_S}{\Lambda_T} \frac{\partial h}{\partial z} = \\ &\frac{k_S}{g}\left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla - \chi \Delta\right)(F - D_\nu f) + \frac{\chi - k_S}{\Lambda_T} \Delta_r f + \\ &Q(k_S - \chi) \frac{\delta(z)\delta(r)}{2\pi r} \theta(t) \\ &Q = \frac{\alpha P}{c_p \rho_0}, \quad D = \Delta - \left(\frac{1}{\Lambda_S} + \frac{1}{\Lambda_T}\right) \frac{\partial}{\partial z}, \quad D_\nu = \Delta\left(\frac{\partial}{\partial t} - \nu \Delta\right) \end{aligned} \tag{1.6}$$

Substituting the expression for h from the first equation of (1.6) into the second equation, we obtain a single equation for the unknown function f, which determines $\mathbf{u}, T', S'$ in accordance with (1.2), (1.4)-(1.6)

$$\begin{aligned} &\left[\left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right) D^{-1} - \chi - k_S\right]\left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right)(F - D_\nu f) + \\ &k_S \chi \Delta (F - D_\nu f) + \left(\frac{\chi}{\Lambda_S} + \frac{k_S}{\Lambda_T}\right) g \Delta_r f - \\ &\left(\frac{1}{\Lambda_S} + \frac{1}{\Lambda_T}\right) g\left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right) D^{-1} \Delta_r f = \\ &gQ\left[k_S \frac{\delta(z)\delta(r)}{2\pi r} \theta(t) + \frac{1}{4\pi}\left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right)(r^2 + z^2)^{-1/2} \theta(t)\right] \end{aligned} \tag{1.7}$$

where D^{-1} is an operator inverse to D.

To simplify the subsequent analysis, we must reduce Eq.(1.7) to its dimensionless form. We choose here, as the scales of the distance d and time τ, the radius of the ascending jet generated by the heat source, and the time of its formation. Thus we must consider the problem of determining d and τ during the initial stages of the process.

2. Determining the spatial and time scales. Remembering that the stratification of the fluid, diffusion of salt and higher-order non-linear terms of the system (1.6) have little effect on the nature of the flow during its initial stages, we can obtain a system describing the process in the first instances after the heat source is switched on

$$\left(\frac{\partial}{\partial t} + \mathbf{w}\cdot\nabla\right) T' = \chi\Delta T' + \frac{Q}{\alpha}\,\frac{\delta(z)\,\delta(r)}{2\pi r}\,\theta(t)$$
$$\Delta T' = \frac{1}{\alpha\chi g}\left(\frac{\partial}{\partial t} + \mathbf{w}\cdot\nabla\right) D_\nu f - \frac{Q}{\alpha\chi}\,\frac{\delta(z)\,\delta(r)}{2\pi r}\,\theta(t) \tag{2.1}$$

Let us determine d and τ as dimensions of the domains of space and time in which the solution of system (2.1) can be represented in the form of a temperature distribution, in the case when (2.1) contains no convection terms, supplemented by the corrections generated by the presence of these convection terms. In accordance with this we shall expand T' in the series

$$T' = T_0' + \varepsilon T_1' + \varepsilon^2 T_2' + \ldots, \quad \varepsilon = Q\chi^{-3/2}g^{1/2} \tag{2.2}$$

where ε is a dimensionless parameter characterizing the contribution of the convection terms of system (2.1) towards the temperature distribution. At the initial instants the contribution will be larger the greater the source strength.

The process of heating the fluid causes buoyancy and leads to the appearance of a convective flow. The convection term has the form

$$\mathbf{w}\cdot\nabla T' = \varepsilon\mathbf{w}_0\cdot\nabla T_0' + \varepsilon^2(\mathbf{w}_0\cdot\nabla T_1' + \mathbf{w}_1\cdot\nabla T_0') + \ldots \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), equating terms of like power in ε and taking (1.4) into account, we obtain a sequence of systems of equations, from which we obtain the terms of the expansion (2.2)

$$\frac{\partial T_0'}{\partial t} = \chi\Delta T_0' + \frac{Q}{\alpha}\,\frac{\delta(z)\,\delta(r)}{2\pi r}\,\theta(t)$$
$$\Delta T_0' = \frac{1}{\alpha\chi g}\,\frac{\partial}{\partial t} D_\nu f_0 - \frac{Q}{\alpha\chi}\,\frac{\delta(z)\,\delta(r)}{2\pi r}\,\theta(t) \tag{2.4}$$
$$\frac{\partial T_n'}{\partial t} + \sum_{i=0}^{n-1}\mathbf{w}_{n-i-1}\cdot\nabla T_i' = \chi\Delta T_n'$$
$$\Delta T_n' = \frac{1}{\alpha\chi g}\,\frac{\partial}{\partial t} D_\nu f_n + \frac{1}{\alpha\chi g}\sum_{i=0}^{n-1}\mathbf{w}_{n-i-1}\cdot\nabla D_\nu f_i, \quad n = 1, 2, \ldots$$

where f_i is the i-th term of the expansion of f in powers ε.

The conditions ensuring the validity of relations (2.2)-(2.4) have the form

$$\varepsilon\left|\frac{T_n'}{T_{n-1}'}\right| \ll 1, \quad \varepsilon\left|\frac{1}{\chi\Delta T_{n-1}'}\sum_{i=0}^{n-1}\mathbf{w}_{n-i-1}\cdot\nabla T_i'\right| \ll 1$$
$$\varepsilon\left|\frac{f_n}{f_{n-1}}\right| \ll 1, \quad \varepsilon\frac{|\mathbf{w}_n|}{|\mathbf{w}_{n-1}|} \ll 1, \quad n = 1, 2, \ldots \tag{2.5}$$

Since ε can take arbitrary values, it follows that the following conditions must hold together with (2.5):

$$\left|\frac{T_n'}{T_{n-1}'}\right| \ll 1, \quad \left|\frac{f_n}{f_{n-1}}\right| \ll 1, \quad n = 1, 2, \ldots \tag{2.6}$$

etc.

The critical values of the distance and time d_* and τ_* for which conditions (2.6) no longer hold, represent the required scales and are found from the relations

$$\left|\frac{T_1'}{T_0'}\right| = 1, \quad \left|\frac{1}{\chi\Delta T_0'}\,\mathbf{w}_0\cdot\nabla T_0'\right| = 1 \tag{2.7}$$

with $(r^2 + z^2)^{1/2} = d_*$, $t = \tau_*$. Using (2.4), and shall write (2.7) in the specific form

$$\frac{\mathbf{w}_0\cdot\nabla T_0'}{\chi\Delta T_0'} \approx gQ\,\frac{\tau_*}{128\pi\chi^2 m^4}\left[\frac{\pi^{1/2}}{2m}\,E(m) + 1\right] = 1 \tag{2.8}$$
$$\frac{T_1'}{T_0'} \approx gQ\,\frac{8\tau_* m^7}{\pi\chi^3 E(m)}\sum_{n=0}^{\infty}\left[\frac{n+3}{(2n+7)!} - \frac{4(n+4)}{(2n+9)!}\right] \times$$

$$\frac{(2m)^{2n}}{2^{1/2}}\frac{\partial^{2n+1}}{\partial\,(2m)^{2n+1}}\left[E\left(\frac{m}{2^{1/2}}\right)-\right.$$
$$\left.\frac{(-1)^n\,(2m)^{2n}}{6\pi^{1/2}}\frac{\partial^n}{\partial p^n}\left(\frac{\operatorname{arctg} p^{1/2}}{p^{1/2}}\right)\Big|_{p=1}\right]=1$$
$$E\,(x)=\exp\,(x^2)\,\operatorname{erfc}\,(x),\quad m={}^1/_2 d_*/(\chi\tau_*)^{1/2}$$

As we have already assumed, at the initial instants the influence of convection terms and corrections to the temperature distribution are larger, the greater the source strength. Here the ratio of the convection to the diffusion terms and the ratio of the correction $\varepsilon T_1'$ to T_0' must be directly proportional to the source strength. But then we find that when $t=\tau_*$ and $(r^2+z^2)^{1/2}=d_*$, the left-hand sides of (2.8) must be independent of the source strength, and this can be arrived at unequivocally if we assume that the quantity m is independent of the source strength and $\tau_*\sim P^{-1}$, whereupon $d_*\sim P^{-1/2}$.

Using the parameters ζ, g and χ of the problem, we can construct quantities with dimensions of time and distance, behaving as P^{-1} and $P^{-1/2}$, as follows:

$$\tau_*=A\,(Q\chi^{-5/3}g^{1/3})^{-1}\,\chi^{1/3}g^{-2/3}$$
$$d_*=B\,(Q\chi^{-5/3}g^{1/3})^{-1/2}\,\chi^{2/3}g^{-1/3} \tag{2.9}$$

Substituting (2.9) into (2.8) and solving the resulting systems for A and B, we obtain $A\approx 10^4$, $B\approx 5\cdot 10^2$.

The numerical computations for $P=4{,}0$ W, $\chi=1{,}5\cdot 10^{-7}$ m^2/sec and $c_p=4\cdot 10^3$ J/kg · K, gave the following results for the radius of the jet and time of its formation: $\tau_*\approx 0{,}1$ sec, $d_*\approx 5\cdot 10^{-4}$ m. These values are close to the experimental values. The computed value of the excess temperature of the fluid $t=\tau_*$ at a distance $d_*/2$ from the source, i.e. in the middle part of the volume of the fluid beginning to move convectively, is equal to $\Delta T'\approx 30$K while experiment gives $\Delta T'\approx 40$ K /6/.

3. An estimate of the critical value of the strength of the heat source. When the convection terms in expansion (2.2) begin to make a major contribution to the temperature distribution as compared with that of diffusion terms, i.e. when ε takes a value of unity, the source strength reaches its critical value, given by the relation

$$P_*=\frac{\chi^{5/3}c_p\rho_0}{\alpha g^{1/3}} \tag{3.1}$$

In an aqueous solution of common salt with a volume concentration of $S_0=5\cdot 10^{-2}$ kg/m^3, $\chi=1{,}5\cdot 10^{-7}$ m^2/sec, $\alpha=2{,}7\cdot 10^{-4}$ K^{-1}, $c_p=4\cdot 10^3$ J/kg · K, $\rho_0=10^3$ kg/m^3, as the critical source strength $P_*\approx 0{,}03$ W, which corresponds to the following value of the global Rayleigh number:

$$\mathrm{Ra}^+=\frac{g\alpha P}{\rho_0 c_p\nu\chi\,(g/\Lambda_S)^{1/2}}\approx 60$$

No convective motion was observed in the experiments for $\mathrm{Ra}^+<60$ /6/

4. Determining the scale of the function f. We choose, as the scale f_* of f, the value of the maximum deviation from zero of the quantity f obtained at the point M of the space, where the sufficient conditions of existence of an extremum hold.

Let us determine f_* at the minimum point, since in this case the fluid flows upwards. The conditions $\partial f/\partial r\,|_M=0$ and $r^{-1}\partial f/\partial r\,|_M>0$ can be satisfied simultaneously only when $r_M=0$. Since the convective motion begins at the instant $t=\tau_*$ at a distance $(r^2+z^2)^{1/2}=d_*$ from the heat source and the radial coordinate of the point M is zero $r_M=0$, we find that $z_M=d_*$.

Taking as f_* the value of the first term of the expansion of the function f at the point M, i.e.

$$f_*=\varepsilon f_0\,(0,d_*,\tau_*),\quad f_0=-g\,\frac{Q}{4\pi}\frac{t^2}{(r^2+z^2)^{1/2}}$$

we obtain

$$f_*=-\frac{A^2}{4\pi B}\,(Q\chi^{5/3}g^{-1/3})^{1/2}$$

where A and B have been found before.

The form of the function f_0 indicates that the heat source generates, at the initial instant, a dipole of velocity sources of intensity $\mu=\chi^{-5/3}g^{4/3}t^2$, independent of the source strength at the instant $t=\tau_*$.

Using the scales obtained, we reduce Eq.(1.7) to the dimensionless form by introducing the dimensionless time t', coordinates r' and z', and function f'

$$t = \tau_* t', \quad r = d_* r', \quad z = d_* z', \quad f = f_* f' \tag{4.1}$$

Substituting (4.1) into (1.7) we obtain (neglecting the primes in the dimensionless variables (4.1))

$$\begin{gathered}\gamma \left[\left(\frac{\partial}{\partial t} + \gamma \mathbf{w}\cdot\nabla\right) D^{-1} - \chi - k_S\right]\left(\frac{\partial}{\partial t} + \gamma \mathbf{w}\cdot\nabla\right)(\gamma F - D_\nu f) - \\ \gamma k_S \chi \Delta (\gamma F - D_\nu f) + \gamma \left(\frac{\chi}{\Lambda_S} + \frac{k_S}{\Lambda_T}\right) g\tau_*^2 \Delta_r f - \\ \gamma \left(\frac{1}{\Lambda_S} + \frac{1}{\Lambda_T}\right) g\tau_*^2 \left(\frac{\partial}{\partial t} + \gamma \mathbf{w}\cdot\nabla\right) D^{-1}\Delta_r f - \\ \frac{\gamma}{4\pi} \operatorname{sign}(P)\, \mathbf{w}\cdot\nabla D^{-1} (r^2 + z^2)^{-1/2}\, \theta(t) \frac{A^3}{B^4} = \\ \operatorname{sign}(P)\left[k_S \frac{\delta(z)\,\delta(r)}{2\pi r}\theta(t) + \frac{\delta(t)}{4\pi}(r^2 + z^2)^{-1/2}\right]\frac{A^3}{B^4} \\ \gamma = -\frac{\varepsilon}{4\pi}\frac{A^3}{B^4}, \quad \chi = \frac{\chi\tau_*}{d_*^2}, \quad k_S = \frac{k_S\tau_*}{d_*^2}, \quad \nu = \frac{\nu\tau_*}{d_*^2}\end{gathered} \tag{4.2}$$

All spatial derivatives in (4.2) are taken over the dimensionless coordinates. The initial and boundary conditions on f are obtained by recalculating the initial and boundary conditions for $\mathbf{u}, p, S', T'$ using the relations connecting the dimensionless and the initial physical variables.

Eq.(4.2) has the following properties.

1°. The solution is not symmetrical about $z = 0$, i.e. the distribution of the salinity and temperature and the velocity field are different in the regions above and below the heat source, and this agrees with experiment /1, 5-7/. The linearized system yields a symmetrical solution /8/.

2°. When the sign of z and the source strength change simultaneously, the solution of (4.2) will not change, but the vertical component of the velocity will change its sign, i.e. if the heat "sink" is turned on, a flow pattern identical to that above the heat source will form below the sink.

3°. The source strength determines directly the degree of non-linearity of the process, and it will therefore affect the flow structure and stability.

Further study of the process by which the flow structure is formed involves investigating how the properties of Eq.(4.2) depend on the source strength.

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